Tutorial Note Jan. 23

Problem 1. (i) Solve $z^2 + 2z + (1-i) = 0$

(ii) Solve $z^4 = -8 - 8\sqrt{3}i$

Solution. (i) $0 = z^2 + 2z + (1-i) = (z+1)^2 - i \Rightarrow (z+1)^2 = i$

$$i^{\frac{1}{2}} = e^{\frac{1}{2}\log i} = e^{\frac{1}{2}(\ln|i| + i\arg(i))} = e^{\frac{1}{2}(0 + (\frac{\pi}{2} + 2k\pi)i)} \quad (k \in \mathbb{Z})$$

$$= e^{(\frac{\pi}{4} + k\pi)i} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \quad \text{or} \quad -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$$

Therefore, $z+1 = \begin{cases} \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \\ -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i \end{cases} \Rightarrow z = \left(-1 + \frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2}i$ or $\left(-1 - \frac{\sqrt{2}}{2}\right) - \frac{\sqrt{2}}{2}i$

Remark. One can prove the quadratic formula

The roots of $az^2 + bz + c = 0$ are $z = \dfrac{-b + (b^2 - 4ac)^{\frac{1}{2}}}{2a}$

by similar "completing the square" technique.

(ii) $z^4 = -8 - 8\sqrt{3}i = 16e^{\frac{4}{3}\pi i} \triangleq z_0$, $|z_0| = 16$, $\arg z_0 = \frac{4}{3}\pi + 2k\pi$ $(k \in \mathbb{Z})$

$$\Rightarrow (-8 - 8\sqrt{3}i)^{\frac{1}{4}} = e^{\frac{1}{4}\log(-8 - 8\sqrt{3}i)} = e^{\frac{1}{4}(\ln|z_0| + i\arg(z_0))}$$

$$= e^{\frac{1}{4}(\ln 16 + i(\frac{4}{3}\pi + 2k\pi))} \quad (k \in \mathbb{Z})$$

$$= e^{\ln 2 + (\frac{1}{3}\pi + \frac{k}{2}\pi)i} \quad (k \in \mathbb{Z})$$

$$= 2e^{\frac{1}{3}\pi i},\ 2e^{\frac{5}{6}\pi i},\ 2e^{\frac{4}{3}\pi i},\ 2e^{\frac{11}{6}\pi i}$$

$$= \pm(\sqrt{3} - i),\ \pm(1 + \sqrt{3}i) \qquad \square$$

Problem 2. (i) Express $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$ in terms of z & $\bar{z}$ where $z = x + iy$

(ii) Express $f(z) = \dfrac{z}{\bar{z}} + \dfrac{\bar{z}}{z}$ in terms of

① Polar coordinate $f(z) = u_1(r, \theta) + i v_1(r, \theta)$

② Rectangular coordinate $f(z) = u_2(x, y) + i v_2(x, y)$

Solution: (i) Notice that $x=\frac{1}{2}(z+\bar{z})$, $y=\frac{1}{2i}(z-\bar{z})$. Thus,

$$\begin{aligned}
f(z) &= \left(\tfrac{1}{2}(z+\bar{z})\right)^2 - \left(\tfrac{1}{2i}(z-\bar{z})\right)^2 - 2\left(\tfrac{1}{2i}(z-\bar{z})\right) \\
&\quad + i\left(2\left(\tfrac{1}{2}(z+\bar{z})\right) - 2\left(\tfrac{1}{2}(z+\bar{z})\right)\left(\tfrac{1}{2i}(z-\bar{z})\right)\right) \\
&= \tfrac{1}{4}(z^2+2z\bar{z}+\bar{z}^2) + \tfrac{1}{4}(z^2-2z\bar{z}+\bar{z}^2) + i(z-\bar{z}) \\
&\quad + i(z+\bar{z}) - \tfrac{1}{2}(z^2-\bar{z}^2) \\
&= \bar{z}^2 + 2iz.
\end{aligned}$$

(ii) ① $z=re^{i\theta}$, $\bar{z}=re^{-i\theta}$

$$\Rightarrow f(z) = \frac{z}{\bar{z}} + \frac{\bar{z}}{z} = \frac{re^{i\theta}}{re^{-i\theta}} + \frac{re^{-i\theta}}{re^{i\theta}} = e^{2i\theta} + e^{-2i\theta} = 2\cos 2\theta$$

$$\Rightarrow u_1(r,\theta) = 2\cos(2\theta), \quad v_1(r,\theta) = 0$$

② Apply $|z|^2 = z\bar{z}$,

$$f(z) = \frac{z^2}{\bar{z}\cdot z} + \frac{\bar{z}^2}{z\bar{z}} = \frac{z^2+\bar{z}^2}{|z|^2} = \frac{(x+iy)^2+(x-iy)^2}{x^2+y^2}$$

$$= \frac{2(x^2-y^2)}{x^2+y^2}$$

$$\Rightarrow u_2(x,y) = \frac{2(x^2-y^2)}{x^2+y^2}, \quad v_2(x,y) = 0$$

□

Problem 3. (i) Assume that $\lim_{z\to z_0} f(z) = w_0$, prove that

$$\lim_{z\to z_0} |f(z)| = |w_0|$$

(ii) Assume that $\exists$ positive number M s.t. $|g(z)| \le M$ in a neighborhood of z_0, and $\lim_{z\to z_0} f(z) = 0$. Prove that

$$\lim_{z\to z_0} f(z)\,g(z) = 0.$$

Solution. Recall the definition of $\lim_{z\to z_0} f(z) = w_0$:

$\forall\ \varepsilon>0$, $\exists\ \delta$ s.t. $\forall\ z$ satisfying $0<|z-z_0|<\delta$, we have
$$|f(z)-w_0|<\varepsilon.$$

(i.) We only need to prove the trianglar inequality
$$\big|\,|f(z)|-|w_0|\,\big| \le |f(z)-w_0| \qquad (*)$$
which is equivalent to prove
$$\mathrm{Max}\left\{ |f(z)|-|w_0|,\ |w_0|-|f(z)| \right\} \le |f(z)-w_0|$$
Applying the trianglar inequality $|z_1+z_2| \le |z_1|+|z_2|$,

Taking $z_1 = f(z)-w_0$, $z_2 = w_0$, one has $|f(z)-w_0| \ge |f(z)|-|w_0|$

Taking $z_1 = w_0 - f(z)$, $z_2 = f(z)$, one has $|f(z)-w_0| \ge |w_0|-|f(z)|$

Therefore, $(*)$ is proved.

(ii) In a neighborhood $\{z, |z-z_0| \le \delta_1\}$, one has
$$|f(z)\,g(z)| \le M\,|f(z)|.$$
Then $\forall \varepsilon>0$, $\exists\ \delta_0 = \min\{\delta, \ \delta_1\}$, s.t. $\forall\ z$ satisfying $0<|z-z_0|<\delta_0$,

we have $|f(z)\,g(z)| < M\varepsilon$, which means $\lim_{z\to z_0} f(z)\,g(z) = 0$.

$\square$

Problem 4. Prove that

(i) $|\sin z|^2 = \sin^2 x + \sinh^2 y$, $\quad z = x+iy$

(ii) $|\sin z| \ge |\sin x|$

Solution. We first claim that $\sin(z_1+z_2) = \sin(z_1)\cos z_2 + \cos z_1 \sin z_2$

This can be checked directly from the definition
$$\sin z = \frac{1}{2i}\left(e^{iz}-e^{-iz}\right), \quad \cos z = \frac{1}{2}\left(e^{iz}+e^{-iz}\right).$$

Then we claim that $i\sinh y = \sin(iy)$, $\cos(iy) = \cosh y$

This can be checked directly from the definition.

One further needs $\cosh^2 y - \sinh^2 y = 1$ and $\sin^2 x + \cos^2 x = 1$

Therefore,

$$\sin z = \sin(x+yi) = \sin x \cos(yi) + \sin(yi)\cos x = \sin x \cosh y + i \sinh y \cos x$$

$$\begin{aligned}\Rightarrow |\sin z|^2 &= \sin^2 x \cosh^2 y + \sinh^2 y \cos^2 x \\ &= \sin^2 x\,(1+\sinh^2 y) + \sinh^2 y \cos^2 x \\ &= \sin^2 x + \sinh^2 y\,(\sin^2 x + \cos^2 x) = \sin^2 x + \sinh^2 y\end{aligned}$$

Therefore, we have proved part (i), and (ii) follows directly from (i). $\square$